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NATURAL VIBRATION OF RECTANGULAR PLATES CONSIDERED AS TRIDIMENSIONAL SOLIDS

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A variational method developed by the authors (named WEM) for one- and two-dimensional boundary value problems is extended to three dimensions. The availability of a classical solution obtained through the inverse method, which is briefly included in an appendix, allows the confirmation of the exactness of the alternative solution herein presented. A prismatic solid supported by shear diaphragms at four consecutive faces is analysed in particular. The numerical values of natural transversal frequencies enable the evaluation of the degree of approximation involved in using Mindlin's theory for thick plates. Also, comparison is made with results from a three-dimensional Ritz formulation.

1. INTRODUCTION

The variational method to be developed in what follows for three-dimensional problems (as an extension of one and two dimensions [1–3]) is based essentially on solving a wide range of differential boundary value problems—ordinary or not, linear or not—by means of "extremizing" a proper functional, and using suitable sequences. The authors have named it WEM (Whole Element Method) since the domain is considered as a single element despite the existence of discontinuities such as intermediate supports, springs, masses, etc.

The variational approach is well-known [4, 5] for linear problems involving positive and symmetric (energetic) functionals and it should be mentioned that such type of functional will be used herein to solve the title problem. The authors have extended the functional pattern to almost any type of boundary value problems. Another innovation consists in the fact that WEM is not a Ritz method as traditionally known. In effect, in the present approach, extended Fourier series are used. The functions to be linearly combined do not satisfy—in general—by themselves the essential boundary conditions (BC) and at the same time belong to a (*a priori*) complete set.

In the present paper the theorems and corollaries justifying the exactness of the proposal, are briefly included and demonstrated. It is worth noting that it has been theoretically shown that the usage of WEM in any differential boundary value problem consists in applying a *pseudo-theorem of virtual work* to the proposed "extremizing" sequences.

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Extensive investigations of the vibration of thick plates have been carried out taking into account various shapes and boundary conditions. A thorough and comprehensive literature survey on this subject was published by Liew *et al*. [6]. It is worth mentioning the important work of Srinivas *et al*. [7, 8] in which three-dimensional theory is used to study the vibration of simply supported homogeneous and laminated thick rectangular plates. Liew *et al*. [9] report a continuum Ritz formulation and comparisons with results from both classical and Mindlin's plate theory. In addition, the authors include numerous sets of deflection contour plots and three-dimensional deformed mode shapes which contribute highly to the understanding of the vibration of these structural elements.

The plate is herein assumed as a rectangular prism of arbitrary aspect ratio with four consecutive sides supported by shear diaphragms (SD). Values of frequency parameters are calculated in case of transverse vibrational modes. Other modes such as *axial-breathing* ones are not considered here though calculation of them offers no inconvenience. The transverse frequencies are compared with an existing classical solution [10] obtained by means of Saint-Venant's inverse method and which is briefly described in Appendix A. It should be noted that these results are coincident with the ones reported in reference [8].

Additionally, a comparison is also made with the solution by Mindlin's theory for thick plane plates [12]. In doing so, one is able to bound the range of validity of this theory, specially regarding the shear factor. The thick simply supported plate is addressed in this paper as a three-dimensional solid with SD, plates with other BC being studied. Also the application of this variational methodology to the rectangular beam in order to compare Bernoulli and Timoshenko beam theories is at present under study.

2. ENERGY FUNCTIONAL

For tridimensional, isotropic, motion problems the classical elasticity yields a differential equations system, known as Lamé equations, in the components of the displacement vector u, v and w (functions of space and time) which is, in rectangular Cartesian co-ordinates (x, y, z) and using classical notation,

$$(\mu/2)\nabla^2 u + (\lambda/2v)\partial I/\partial x = \rho\, \partial^2 u/\partial t^2 + F_x^*,$$
$$(\mu/2)\nabla^2 v + (\lambda/2v)\, \partial I/\partial y = \rho\, \partial^2 v/\partial t^2 + F_y^*,$$
$$(\mu/2)\nabla^2 w + (\lambda/2v)\, \partial I/\partial z = \rho\, \partial^2 w/\partial t^2 + F_z^*, \tag{1}$$

where ρ is the mass density of the body at the considered point; $\mu = E/(1+v)$ and $\lambda = \mu v/(1-2v)$ are Lamé constants; v is Poisson's coefficient; E is the modulus of elasticity; F_x^*, F_y^* and F_z^* are functions proportional to the mass force components; $I = \epsilon_x + \epsilon_y + \epsilon_z$ is the linear strain invariant, ϵ_x, ϵ_y and ϵ_z are the specific axial strains and t is the temporal co-ordinate. Admitting normal modes of vibration with circular frequency ω, i.e., $F_x^* = F_y^* = F_z^* = 0$ and

$$u = \hat{u}(x, y, z, t) = u(x, y, z, t)\cos\omega t, \qquad v = \hat{v}(x, y, z, t) = v(x, y, z, t)\cos\omega t,$$
$$w = \hat{w}(x, y, z, t) = w(x, y, z, t)\cos\omega t, \tag{2}$$

the following system of equations is obtained:

$$(\mu/2)\nabla^2 u + (\lambda/2v)\, \partial I/\partial x - \rho\omega^2 u = 0, \qquad (\mu/2)\nabla^2 v + (\lambda/2v)\, \partial I/\partial y - \rho\omega^2 v = 0,$$
$$(\mu/2)\nabla^2 w + (\lambda/2v)\, \partial I/\partial z - \rho\omega^2 w = 0. \tag{3}$$

The energy functional $\mathscr{F}$ corresponding to equation (3) (three-dimensional strain energy) is easily obtained as

$$\mathscr{F} \equiv \mathscr{F}[u, v, w] = (\lambda/2)\|\epsilon_x + \epsilon_y + \epsilon_z\|^2 + (\mu/4)(\|\epsilon_x\|^2 + \|\epsilon_y\|^2 + \|\epsilon_z\|^2) + \mu(\|\gamma_{xy}\|^2 + \|\gamma_{yz}\|^2 + \|\gamma_{zx}\|^2) - (\rho\omega^2/2)(\|u\|^2 + \|v\|^2 + \|w\|^2), \quad (4)$$

where by means of the kinematic relationships it is known that

$$\epsilon_x = \partial u/\partial x, \quad \epsilon_y = \partial v/\partial y, \quad \epsilon_z = \partial w/\partial z, \quad \gamma_{xy} = \gamma_{yx} = \partial u/\partial y + \partial v/\partial x,$$
$$\gamma_{yz} = \gamma_{zy} = \partial v/\partial z + \partial w/\partial y, \quad \gamma_{zx} = \gamma_{xz} = \partial w/\partial x + \partial u/\partial z. \quad (5)$$

In equation (4), functional analysis notation has been introduced. If $P = P(x, y, z)$ and $Q = Q(x, y, z)$ are two square integrable functions in the domain D of interest, the following definitions apply:

$$(P, Q) \equiv \int\int\int P(p, q, r)Q(p, q, r)\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}r;$$

$$\|P\|^2 \equiv \int\int\int P^2(p, q, r)\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}r < \infty. \quad (6)$$

3. "EXTREMIZING" SEQUENCES

Since in this problem the functional $\mathscr{F}$ is symmetrical and positive definite [5] the sequences will be, strictly speaking, minimizing. The Fourier series for rectangular, prismatic domains $\{D: 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1,\ 0 \leqslant z \leqslant 1\}$ which can be used are of the form $\Sigma\Sigma\Sigma\ f_1 f_2 f_3$ where $f_1 f_2 f_3$ is any of the following combinations:

$$s_i s_j s_k, \quad s_i s_j c_k, \quad s_i c_j s_k, \quad s_i c_j c_k, \quad c_i s_j s_k, \quad c_i s_j c_k, \quad c_i c_j s_k, \quad c_i c_j c_k, \quad (7)$$

where the following notation has been used: $\alpha_i \equiv i\pi$, $\beta_j \equiv j\pi$, $\gamma_k \equiv k\pi$, $s_i \equiv \sin \alpha_i x$, $s_j \equiv \sin \beta_j y$, $s_k \equiv \sin \gamma_k z$, $c_i \equiv \cos \alpha_i x$, $c_j \equiv \cos \beta_j y$, $c_k \equiv \cos \gamma_k z$, $(i, j, k = 0, 1, 2, \ldots)$.

Such series guarantee, as is known, the convergence in the mean of any square integrable function. However, the methodology herein proposed requires also the uniform convergence of the so-called continuous essential functions, which in this problem are **u**, **v** and **w**. In what follows it will be briefly shown how those tridimensional series are generated to represent a continuous function.

By starting from a continuous function $\varphi = \varphi(x, y, z)$ of three variables for which one requires uniform convergence in D, the following two expansions are chosen as an example of possible sequences:

$$\varphi_M(x, y, z) = \sum_i^M B_i(y, z) \sin \alpha_i x + x B_0(y, z) + b_0(y, z),$$

$$\varphi_M(x, y, z) = \sum_i^M A_i(y, z) \cos \alpha_i x + A_0(y, z), \quad (8)$$

from which, it will suffice, in order to achieve uniform convergence, that (Fourier theory):

$$B_i(y, z) = 2 \int_0^1 \varphi(\eta, y, z) \sin \alpha_i \eta\,\mathrm{d}\eta, \quad (9)$$

$$B_0(y, z) = \varphi(1, y, z) - \varphi(0, y, z); \qquad b_0(y, z) = \varphi(0, y, z) \quad (10)$$

Instead,

$$A_i(y, z) = 2\int_0^1 \varphi(\eta, y, z)\cos\alpha_i\eta\,\mathrm{d}\eta, \qquad A_0(y, z) = \int_0^1 \varphi(\eta, y, z)\,\mathrm{d}\eta. \tag{11}$$

Equations (8), with the support function $xB_0(y, z) + b_0(y, z)$ give, as can be easily demonstrated, uniform convergence for φ. Now, if any of the functions of (y, z) involved in equations (8) are expanded in an analogous form in the variable y, and then and in a similar fashion, the functions of z, one obtains all possible combinations.

By selecting the mode shapes, corresponding to the transversal (bending) type, of a rectangular prism with shear diaphragms in $x = 0{,}1$ and $y = 0{,}1$ (see Figure 1), the essential boundary conditions of the problem are

$$x = 0, 1: \quad w = 0, v = 0; \qquad y = 0, 1: \quad w = 0, u = 0. \tag{12}$$

Non-dimensionalized co-ordinates have been used. Additionally, it is required that

$$w(x, y, z) = w(x, y, 1 - z), \quad u(x, y, z) = -u(x, y, 1 - z), \quad v(x, y, z) = -v(x, y, 1-z). \tag{13}$$

In such a way the tridimensional, complete, Fourier series generated as explained at the beginning of the section, and yielding uniform convergence for u, v and w, are reduced to

$$u_{LMN}(x, y, z) = \sum_i^L\sum_j^M\sum_k^N A_{ijk}c_is_jc_k + \sum_j^M\sum_k^N A_{0jk}s_jc_k,$$

$$v_{LMN}(x, y, z) = \sum_i^L\sum_j^M\sum_k^N B_{ijk}s_ic_jc_k + \sum_i^L\sum_k^N B_{i0k}s_ic_k,$$

$$w_{LMN}(x, y, z) = \sum_i^L\sum_j^M\sum_k^N C_{ijk}s_is_js_k + \sum_i^L\sum_j^M d_{ij}s_is_j, \tag{14}$$

where k should be odd ($k = 1, 3, 5, \ldots$).

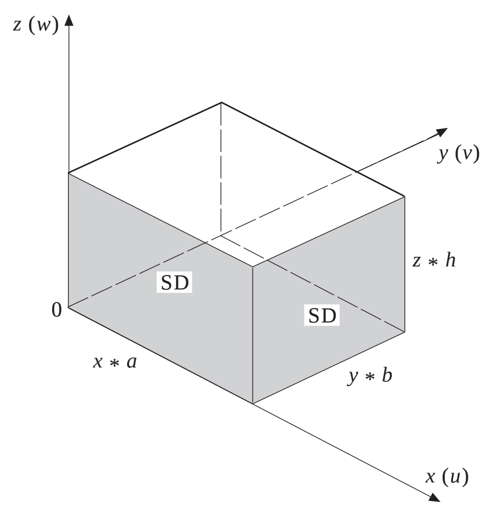

Figure 1. Prismatic solid supported by shear diaphragms on faces (x, z) ($y = 0$ and $y = b$) and (y, z) ($x = 0$ and $x = a$).

4. THEOREMS, COROLLARY AND APPLICATION

The selected sequences give uniform convergence for **u**, **v** and **w** which are assumed continuous. But for some derivatives of these functions only convergence in the mean (in L_2) is achieved. However, this will be enough to demonstrate that the sought frequencies converge to the exact value. Then, one has, at least,

$$\|\partial\Delta\mathbf{u}/\partial x_m\| \equiv \|\partial\mathbf{u}_{LMN}/\partial x_m - \partial\mathbf{u}/\partial x_m\| \to 0, \qquad L, M, N \to \infty,$$
$$\|\partial\Delta\mathbf{v}/\partial x_m\| \equiv \|\partial\mathbf{v}_{LMN}/\partial x_m - \partial v/\partial x_m\| \to 0, \qquad L, M, N \to \infty,$$
$$\|\partial\Delta\mathbf{w}/\partial x_m\| \equiv \|\partial\mathbf{w}_{LMN}/\partial x_m - \partial w/\partial x_m\| \to 0, \qquad L, M, N \to \infty, \tag{15}$$

where $m = 1, 2, 3$ and $x_1 = x,\ x_2 = y,\ x_3 = z$ and also

$$|\Delta\mathbf{u}| \equiv |\mathbf{u}_{LMN} - \mathbf{u}| \to 0, \qquad L, M, N \to \infty, \qquad \forall x, y, z;$$
$$|\Delta\mathbf{v}| \equiv |\mathbf{v}_{LMN} - \mathbf{v}| \to 0, \qquad L, M, N \to \infty, \qquad \forall x, y, z;$$
$$|\Delta\mathbf{w}| \equiv |\mathbf{w}_{LMN} - \mathbf{w}| \to 0, \qquad L, M, N \to \infty, \qquad \forall x, y, z. \tag{16}$$

As was mentioned before, although this is not the case, the derivatives of u, v and w are required to be square integrables and not necessarily continuous.

Theorem 1. The functional $\mathscr{F} \equiv \mathscr{F}[u, v, w]$ is an extreme among $\mathscr{F}_{LMN} \equiv \mathscr{F}[u_{LMN}, v_{LMN}, w_{LMN}]$, where u_{LMN}, v_{LMN} and w_{LMN} are "extremizing" sequences.

It should be noted that u, v and w are not required, for the time being, to satisfy the differential system (3).

Demonstration: based upon definition (4):

$$\begin{aligned}\Delta\mathscr{F} \equiv \mathscr{F}_{LMN} - \mathscr{F} = {} & (\lambda/2)[\|\Delta I\|^2 + 2(I, \Delta I)] + (\mu/4)[\|\Delta\epsilon_x\|^2 + \|\Delta\epsilon_y\|^2 + \|\Delta\epsilon_z\|^2 \\ & + 2(\epsilon_x, \Delta\epsilon_x) + 2(\epsilon_y, \Delta\epsilon_y) + 2(\epsilon_z, \Delta\epsilon_z)] + \mu[\|\Delta\gamma_{xy}\|^2 + \|\Delta\gamma_{xz}\|^2 \\ & + \|\Delta\gamma_{yz}\|^2 + 2(\gamma_{xy}, \Delta\gamma_{xy}) + 2(\gamma_{xz}, \Delta\gamma_{xz}) + 2(\gamma_{yz}, \Delta\gamma_{yz})] \\ & - (\rho\omega^2/2)[\|\Delta u\|^2 + \|\Delta v\|^2 + \|\Delta w\|^2 + 2(u, \Delta u) + 2(v, \Delta v) + 2(w, \Delta w)],\end{aligned} \tag{17}$$

where

$$\Delta I \equiv (I_{LMN} - I) = (\partial u_{LMN}/\partial x + \partial v_{LMN}/\partial y + \partial w_{LMN}/\partial z) - I,$$
$$\Delta\epsilon_x \equiv \epsilon_{x_{LMN}} - \epsilon_x = \partial u_{LMN}/\partial x - \epsilon_x, \text{ etc.,}$$
$$\Delta\gamma_{xy} \equiv \gamma_{xy_{LMN}} - \gamma_{xy} = \left(\frac{\partial u_{LMN}}{\partial y} + \frac{\partial v_{LMN}}{\partial x}\right) - \gamma_{xy}, \text{ etc.} \tag{18}$$

Now, recalling Hooke's law for isotropic materials,

$$\sigma_x = \mu\epsilon_x + \lambda I, \qquad \sigma_y = \mu\epsilon_y + \lambda I, \qquad \sigma_z = \mu\epsilon_z + \lambda I, \qquad \tau_{xy} = \tau_{yx} = (\mu/2)\gamma_{xy} = (\mu/2)\gamma_{yx},$$
$$\tau_{yz} = \tau_{zy} = (\mu/2)\gamma_{yz} = (\mu/2)\gamma_{zy}, \qquad \tau_{xz} = \tau_{zx} = (\mu/2)\gamma_{xz} = (\mu/2)\gamma_{zx}, \tag{19}$$

the Cauchy–Schwartz theorem,

$$(P,Q) \leqslant \|P\|\|Q\|, \tag{20}$$

and after making use of the divergence theorem in order to integrate by parts, it can be observed that according to equation (15) and (16) the following statement is verified:

$$|\Delta\mathscr{F}| = 0, \text{ for } L, M, N \to \infty. \tag{21}$$

That is,

$$\mathscr{F}_{LMN} = \mathscr{F}, \text{ for } L, M, N \to \infty. \tag{22}$$

Corollary: it has been demonstrated that $\mathscr{F}$ is an extreme among $\mathscr{F}_{LMN}$ but also equation (22) indicates that $\mathscr{F}_{LMN}$ is an extreme with an adequate selection of the sequence constants.

Theorem 2. In order for $\mathscr{F}$ to be an extreme among the functionals $\mathscr{F}_{LMN}$ being u_{LMN}, v_{LMN}, w_{LMN} sequences (not necessarily "extremizing") which satisfy the eventual essential boundary conditions (i.e., those involving u, v and w) then the functions u, v and w must satisfy the differential system (3).

Demonstration: let one state $\mathscr{F}^* \equiv \mathscr{F}[u + pu_{LMN}, v + qv_{LMN}, w + rw_{LMN}]$ (p, q and $r \in \Re$) and $\bar{\delta}\mathscr{F}$ that is,

$$\bar{\delta}\mathscr{F} \equiv \partial\mathscr{F}^*/\partial p + \partial\mathscr{F}^*/\partial q + \partial\mathscr{F}^*/\partial r|_{p=q=r=0} = 0, \tag{23}$$

or which is equivalent, the three simultaneous conditions:

$$\partial\mathscr{F}^*/\partial p|_{p=q=r=0} = 0, \qquad \partial\mathscr{F}^*/\partial q|_{p=q=r=0} = 0, \qquad \partial\mathscr{F}^*/\partial r|_{p=q=r=0} = 0. \tag{24}$$

If the divergence theorem is applied, one concludes, by the first of equations (24), that the following should be satisfied:

$$\{[(\mu(\partial\epsilon_x/\partial x) + \lambda(\partial I/\partial x)) + (\mu/2)(\partial\gamma_{xy}/\partial y + \partial\gamma_{xz}/\partial z) + \rho\omega^2 u], u_{LMN}\} = 0,$$

$$\iint\left[(\mu\epsilon_x + \lambda I)l + \frac{\mu}{2}(\gamma_{xy}m + \gamma_{xz}n)\right]u_{LMN}\,\mathrm{d}\Omega = 0, \tag{25}$$

where l, m and n are the cosines of the normal to the surface surrounding the domain. The other two conditions of equations (24) are worked out analogously. As can be seen, accepting the equilibrium equations (1) and that the sequences satisfy the eventual nullity conditions, theorem 2 is demonstrated.

4.1. APPLICATION PROCEDURE

When stating the extreme condition for $\mathscr{F}_{LMN}$ (according to the previous corollary) one obtains

$$\delta\mathscr{F}_{LMN} = 0, \tag{26}$$

where δ denotes variation w.r.t. the sequence constants. The following expression results:

$$\begin{aligned}\lambda(I_{LMN}, \delta I_{LMN}) &+ \mu[(\epsilon_{x_{LMN}}, \delta\epsilon_{x_{LMN}}) + (\epsilon_{y_{LMN}}, \delta\epsilon_{y_{LMN}}) + (\epsilon_{z_{LMN}}, \delta\epsilon_{z_{LMN}})] + (\mu/2)[(\gamma_{xy_{LMN}}, \delta\gamma_{xy_{LMN}}) \\ &+ (\gamma_{yz_{LMN}}, \delta\gamma_{yz_{LMN}}) + (\gamma_{xz_{LMN}}, \delta\gamma_{xz_{LMN}})] - \rho\omega^2[(u_{LMN}, \delta u_{LMN}) + (v_{LMN}, \delta v_{LMN}) \\ &+ (w_{LMN}, \delta w_{LMN})] = 0.\end{aligned} \tag{27}$$

This is no more than a *pseudo-theorem of virtual work* applied to the "extremizing" sequences.

5. RESULTS

After applying equation (27) and factoring (according to equations (14)), $\sum_i\sum_j\sum_k \delta A_{ijk}$, $\sum_i\sum_j\sum_k \delta B_{ijk}$, $\sum_i\sum_j\sum_k \delta C_{ijk}$, $\sum_j\sum_k \delta A_{0jk}$, $\sum_i\sum_k \delta B_{i0k}$, $\sum_i\sum_j \delta d_{ij}$, the following equations result:

$$(\lambda+2G)\left(\frac{\alpha_i^2}{a^2}A_{ijk}\right)+\lambda\left(\frac{\alpha_i\beta_j}{ab}B_{ijk}-\frac{\alpha_i\gamma_k}{ah}C_{ijk}\right)+G\left[-\frac{\gamma_k}{h}\left(\frac{\alpha_i}{a}C_{ijk}-\frac{\gamma_k}{h}A_{ijk}+\frac{2\alpha_i}{a}d_{ij}I_k\right)\right.$$

$$\left.+\frac{\beta_j}{b}\left(\frac{\beta_j A_{ijk}}{b}+\frac{\alpha_i B_{ijk}}{a}\right)\right]-\Omega^{*2}A_{ijk}=0, \tag{28}$$

$$(\lambda+2G)\left(\frac{\beta_j^2}{b^2}B_{ijk}\right)+\lambda\left(\frac{\alpha_i\beta_j}{ab}A_{ijk}-\frac{\beta_j\gamma_k}{bh}C_{ijk}\right)+G\left[-\frac{\gamma_k}{h}\left(\frac{\beta_j}{b}C_{ijk}-\frac{\gamma_k}{h}B_{ijk}+\frac{2\beta_j}{b}d_{ij}I_k\right)\right.$$

$$\left.+\frac{\alpha_i}{a}\left(\frac{\beta_j A_{ijk}}{b}+\frac{\alpha_i B_{ijk}}{a}\right)\right]-\Omega^{*2}B_{ijk}=0, \tag{29}$$

$$(\lambda+2G)\left(\frac{\gamma_k^2}{h^2}C_{ijk}\right)-\lambda\left(\frac{\alpha_i\gamma_k}{ah}A_{ijk}+\frac{\beta_j\gamma_k}{bh}B_{ijk}\right)+G\left[\frac{\beta_j}{b}\left(\frac{\beta_j}{b}C_{ijk}-\frac{\gamma_k}{h}B_{ijk}+\frac{2\beta_j}{b}d_{ij}I_k\right)\right.$$

$$\left.+\frac{\alpha_i}{a}\left(\frac{\alpha_i C_{ijk}}{a}-\frac{\gamma_k A_{ijk}}{h}+\frac{2\alpha_i}{a}d_{ij}I_k\right)\right]-\Omega^{*2}[C_{ijk}+2d_{ij}I_k]=0, \tag{30}$$

$$G\left[\frac{\beta_j^2}{b^2}d_{ij}+\frac{\beta_j^2}{b^2}\sum_l C_{ijl}I_l-\frac{\beta_j}{bh}\sum_l\gamma_{ijl}I_l+\frac{\alpha_i^2}{a^2}d_{ij}+\frac{\alpha_i^2}{a^2}\sum_l C_{ijl}I_l-\frac{\alpha_i}{ah}\sum_l\gamma_l A_{ijl}I_l\right]$$

$$-\Omega^{*2}\left[d_{ij}+\sum_l C_{ijl}I_l\right]=0, \tag{31}$$

$$A_{0jk}[(r^2\beta_j^2+R^2\gamma_k^2)-\Omega^{*2}/G]=0, \qquad B_{i0k}[(\alpha_i^2+R^2\gamma_k^2)-\Omega^{*2}/G]=0, \tag{32, 33}$$

in which $\Omega^{*2}\equiv\rho\omega^2$, $I_q=(1,\sin qz)$. The non-dimensionalization has been carried out using the parameters

$$D=Eh^3/12(1-\nu^2), \qquad m=(1-2\nu)/12\nu(1-\nu), \qquad \Omega=\sqrt{(\gamma/D)}\omega a^2,$$

$$k^*=(1-2\nu)/\nu, \qquad r=a/b, \qquad G=\mu/2, \qquad R=a/h, \qquad \gamma=\rho h.$$

From equations (32) and (33) and for $A_{0jk}\neq 0$, $B_{i0k}\neq 0$ and the resting coefficients with null value, the frequency parameters arise as

$$\Omega_{jk}^2=(2k^*R^2/m)(r^2\beta_j^2+R^2\gamma_k^2), \qquad \Omega_{ik}^2=(2k^*R^2/m)(\alpha_j^2+R^2\gamma_k^2). \tag{34}$$

These special "pure shear"-like mode shapes result with $i=1,2,3,\dots;$ $j=1,2,3,\dots;k=odd$, and according with equation (14) for the frequencies (Ω_{jk}) as

$$u_{MN}^{jk}=A_{0jk}s_jc_k, \qquad v_{LMN}=w_{LMN}=0, \tag{35}$$

and for the frequencies Ω_{ik} as

$$v_{LN}^{ik}=B_{i0k}s_ic_k, \qquad w_{LMN}=u_{LMN}=0. \tag{36}$$

On the other hand, numerical results of the transverse frequency parameter Ω are obtained by means of the procedure described in this section, assuming $A_{0jk}=B_{i0k}=0$ and working out the equations (28) to (31). The values are compared with a classical solution included in Appendix A and also with results reported in reference [12] obtained using Mindlin's theory. Table 1 depicts values of the frequency parameter for the case of square prism ($r = a/b = 1$) corresponding to the first mode of vibration ($i = j = 1$) with $R = a/h = 1, 10, 25, 45, 250$ being $\nu = 1/3$. The results obtained using the classical solution (Appendix A) are compared with values from WEM using 100 000 terms in the sums. Also, exact values in all the digits are shown. They were found by increasing the number of terms (not necessarily the same for all columns) until the desired accuracy was achieved. It should be noted that a larger number of terms is required to meet the goal when $R = a/b$ is increased (approaching thin plates). Notwithstanding, this is a numerical obstacle since it may be theoretically shown that in the limit for $R \to \infty$ the results are coincident with the Germain–Lagrange theory frequencies [13]. The authors have developed an algorithm for thin rectangular plates that is more appropriate (large R) [11]. Despite the fact that WEM yields exact values and the numerical algorithm automatically gives the desired exact digits, a convergence study is included in Table 2.

Also, a prism of ratio $r = a/b = 2/3$ is solved for $R = a/h = 2{\cdot}5, 250$ (Table 3). (Shown frequencies are not necessarily the lower ones). In the same table the limit case of $r = a/b \to 0$ is presented. It should be noted that $R \to \infty$ indicates thin plates. The number of terms is indicated for each thickness. Again when dealing with thin plates a large number of terms is needed to attain the exact frequencies.

Tables 4 and 5 show a comparison between the method herein presented and results from reference [12] in which Mindlin's theory is used to solve thick plates with $\nu = 0{\cdot}3$ and a shear coefficient of 0·823. Also, results from a continuum three-dimensional Ritz formulation [14] are shown. It should be mentioned that only flexional-type modes are included in the table. Table 4 depicts the values of the frequency parameters corresponding to the first six transversal modes of prisms with $r = a/b = 1$, $R = 5, 10$. As is expected, modes 2 and 3 correspond to the same frequency value. The same for modes 5 and 6. The values corresponding to prisms with $r = 0{\cdot}5, 2/3$ are shown in Table 5. A number of terms ranging from 10^5 to 10^6 has been used.

TABLE 1

Transversal frequency parameter Ω of a prism with $r = a/b = 1$; $\nu = 1/3$; $i = j = 1$. Comparison with classical solution (Appendix A).

	$R = a/h$				
Solution	1	10	25	45	250
WEM (100 000 terms)	7·5466	19·070	19·631	19·720	20·238
WEM (exact values)	7·5466	19·069	19·626	19·704	19·738
Classical	7·5466	19·0694	19·6261	19·7040	19·7380

TABLE 2

Convergence study for the transversal frequency parameter Ω of a prism with $r = a/b = 1$, $R = a/h = 10$, $\nu = 1/3$, $i = j = 1$. Classical solution (Appendix A): $\Omega = 19{\cdot}069$.

Number of terms	250	500	10^3	10^4	10^5	10^6	$2{\cdot}10^6$
Frequency parameter	19·392	19·231	19·151	19·078	19·070	19·069	19·069

TABLE 3

Transversal frequency parameter Ω of a prism with $r = a/b = 2/3$, 0; $\nu = 1/3$. Comparison with classical solution (Appendix A).

Solution	$r = a/b = 2/3$		$r = a/b \to 0$	
	$R = 2{\cdot}5$, $i = 1, j = 2$	$R = 250$ $i = 2, j = 1$	$R = 2{\cdot}5$, $i = 1, j = 1$	$R = 250$, $i = 2, j = 1$
WEM (number of terms)	17·6529(10^5)	43·8643(10^7)	7·9557(10^5)	39·4788(10^7)
Classical	17·6529	43·8593	7·9557	39·4738

TABLE 4

Comparison of transversal frequency parameters Ω of a prism with $r = a/b = 1$, $\nu = 0{\cdot}3$

		$R = a/h$					
		5			10		
Mode	(i, j)	WEM	Ref. [12]	Ref. [14]	WEM	Ref. [12]	Ref. [14]
1	(1, 1)	17·5261	17·4307	17·5264	19·0901	19·0592	19·0898
2, 3	(1, 2), (2, 1)	38·4827	38·0769	38·4826*	45·6193	45·4495	45·6193
4	(2, 2)	55·7872	55·0101	55·7869	70·1042	69·7209	70·1038
5, 6	(1, 3), (3, 1)	65·9960	64·9617	—	85·4876	84·9329	85·4875

*Corresponds to another mode shape in reference [13].

TABLE 5

Transversal frequency parameter Ω of a prism with $R = a/h = 10$, $\nu = 0{\cdot}3$. Comparison with Mindlin's theory

	$r = a/b$					
	0·5			2/3		
Mode	(i, j)	WEM	Ref. [12]	(i, j)	WEM	Ref. [12]
1	(1, 1)	12·0777	12·0646	(1, 1)	13·9118	13·8954
2	(1, 2)	19·0901	19·0592	(1, 2)	26·1917	26·1337
3	(1, 3)	30·4235	30·3461	(2, 1)	40·8747	40·7378
4	(2, 1)	39·1970	39·0708	(1, 3)	45·6194	45·4495
5	(2, 2)	45·6194	45·4495	(2, 2)	52·1456	51·9269
6	(1, 4)	45·6194	45·4495	(2, 3)	70·1042	69·7208

6. CONCLUSIONS

In this work the natural frequencies of a prism supported by shear diaphragms are found by means of two exact solutions. Both are compared with the Mindlin theory results. The exact solution developed in the main part of this work is founded on a variational method which, in short, reduces the algorithm to the evaluation of summations in z. The smaller the thickness h w.r.t. a (large values of R) the more terms should be taken in order to achieve good accuracy. It is not included but can be simply verified, that in the limit as $R \to \infty$ the exact solutions are coincident with the frequency values given by the Germain–Lagrange theory.

The shear diaphragms restraints imposed in the present work prevent the displacements of these diaphragms in the corresponding planes (see equation (12)). When selecting (see equation (13)) the ''flexional''-like mode shapes (k = odd), subsets of the whole set of possible mode shapes are considered. Such subsets are also complete in L_2 from which the exactness of the frequencies and modal shapes depends.

On the other hand, Liew *et al*. [13] have taken into account modes caused by other types of constraints which are not allowed in the present work for the reasons explained above. Obviously, these modes could be considered in a simple way. Additionally it is worth mentioning that due to equation (13) the breathing modes are not calculated either. At last, it is important to point out that the subset considered herein ''catches'' pure shear like-mode shapes (equations (35) and (36)) that in the work of Liew *et al*. [13] are not presented. Obtaining the exact solution by means of this variational method obviously allows one to correct shear coefficients used in Mindlin's theory.

The authors are extending the methodology to prisms with arbitrary boundary conditions. The classical solution is not available in these cases but can be approached using the methodology presented here.

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APPENDIX A: CLASSICAL SOLUTION

The problem of the natural vibrations of a prismatic body is herein addressed with a classical solution in the sense of the inverse method of Saint–Venant. More details may be found in reference [10].

The following functions of x, y, z are proposed to solve the governing equations (3):

$$u_{MN}(x, y, z) = \sum_{j=1}^{N} \sum_{i=1}^{M} A_{ij}(z) \cos \alpha_i x \sin \beta_j y,$$

$$v_{MN}(x, y, z) = \sum_{j=1}^{N} \sum_{i=1}^{M} B_{ij}(z) \sin \alpha_i x \cos \beta_j y,$$

$$w_{MN}(x, y, z) = \sum_{j=1}^{N} \sum_{i=1}^{M} C_{ij}(z) \sin \alpha_i x \sin \beta_j y, \tag{37}$$

where $\alpha_i = i\pi$, $\beta_j = j\pi$ and which satisfy the boundary conditions

$$\sigma_x = v = w = 0, \text{ at } x = 0, 1; \qquad \sigma_y = u = w = 0, \text{ at } y = 0, 1; \tag{38}$$

where σ_x, σ_y are components of the stress tensor. After substituting equation (37) in equations (3) the following system of equations in $A(z) = A_{ij}(z)$, $B(z) = B_{ij}(z)$, $C(z) = C_{ij}(z)$ results, for each pair of values, i, j:

$$\{-[i^2\pi^2 t + d]A + R^2\bar{\bar{A}}\} - ij\pi^2 rtB + i\pi tR\bar{C} = 0,$$

$$-ij\pi^2 rtA + \{-[j^2\pi^2 r^2 t + d]B + R^2\bar{\bar{B}}\} + j\pi trR\bar{C} = 0,$$

$$i\pi tR\bar{A} + j\pi rtR\bar{B} + \{dC - (1 + t)R^2\bar{\bar{C}}\} = 0. \tag{39}$$

The following notation has been used:

$$r = a/b, \qquad R = a/h, \qquad \gamma = \rho h, \qquad D = Eh^3/12(1 - v^2),$$

$$\Omega = \sqrt{\gamma/D}\,\omega a^2, \qquad \hat{\Omega}_{ij} = \pi^2[i^2 + (rj)^2], \qquad Q = \Omega/s, \qquad d = \hat{\Omega}_{ij}(1 - Q),$$

$$s = 2R\hat{\Omega}_{ij}^2\sqrt{\tfrac{3}{2}(1 - v)}, \qquad t = 1/(1 - 2v), \qquad (\bar{\ .\ }) = \partial(\ .\ )/\partial z, \text{ etc.}$$

Note that $\hat{\Omega}_{ij}$ are the frequency parameter values of the thin plate (Germain–Lagrange) theory. Now, the following functions of z are proposed as solution: $A(z) = Fe^{\theta z}$; $B(z) = Me^{\theta z}$; $C(z) = He^{\theta z}$. F, M, H are arbitrary constants and θ the eigenvalues. Then equations (39) become

$$[\theta^2 R^2 - ti^2\pi^2 + d]F - ij\pi^3 rtM + i\pi tR\theta H = 0,$$

$$ij\pi^2 rtF + [\theta^2 R^2 - tj^2\pi^2 r^2 + d]M + j\pi tR\theta H = 0,$$

$$i\pi tR\theta F + j\pi trR\theta M - [\theta^2 R^2(1 + t) + d]H = 0. \tag{40}$$

Six roots (not all distinct) are obtained from stating the nullity condition of the determinant, that is,

$$\Delta_{1,3} = \pm\sqrt{(d + t\hat{\Omega}_{ij}^2)/(1 + t)}, \qquad \Delta_{2,4} = \pm\sqrt{d}; \qquad \Delta_{5,6} = \pm\sqrt{d}, \tag{41}$$

in which $\Delta \equiv \theta a$. After replacing these roots in the proposed functions of z, the general solution of the problem is obtained.

Finally, the boundary conditions for the planes $z = 0$ and $z = h$ should be imposed, i.e.,

$$\tau_{zx} = 0, \qquad \tau_{zy} = 0, \qquad \sigma_z = 0. \tag{42}$$

After some algebraic steps the characteristic equation for the transversal mode of vibration of a prismatic body, yields

$$U/V = 1, \qquad U = \tanh \Delta_1/R/\tanh \Delta_2/R,$$
$$V = \sqrt{[(1 - Q^2)(1 - Q^2/(1 + k))]}/(1 - 0{\cdot}5Q^2)^2\,, \tag{43}$$

from which the frequency parameters Ω are evaluated.